Relation between complex and real version of Fourier transform

Let $f: \mathbb{R} \to \mathbb{R}$ be real-valued signal $\quad (f \in L^2(\mathbb{R}))$

Then

[FMP, Eq. 2.16] $\quad 1)\ f(t) = \int_{\omega \geq 0} d_\omega \sqrt{2} \cos(2\pi(\omega t - \varphi_\omega))\, d\omega$

[FMP, Eq. 2.17] $\quad 2)\ f(t) = \int_{\omega \in \mathbb{R}} c_\omega \exp(2\pi i \omega t)\, d\omega$

with

[FMP, Eq. 2.4] Eq. 2.14 $\quad a)\ d_\omega = \sqrt{2}\, |c_\omega| = \max_{\varphi \in [0,1]} \left(\int_t f(t) \cos_{\omega,\varphi}(t)\, dt \right)$

[FMP, Eq. 2.5] Eq. 2.15 $\quad b)\ \varphi_\omega = -\frac{1}{2\pi} \gamma_\omega = \operatorname{argmax}_{\varphi \in [0,1]} \left(\int_t f(t) \cos_{\omega,\varphi}(t)\, dt \right)$

$$\cos_{\omega,\varphi}(t) = \sqrt{2} \cos(2\pi(\omega t - \varphi))$$

$c)\ c_\omega = \hat{f}(\omega) = \int_{t \in \mathbb{R}} f(t) \exp(-2\pi i \omega t)\, dt$

$$= |c_\omega| \cdot \exp(i \gamma_\omega)$$

Proof: $\quad 2) = \int_{\omega \in \mathbb{R}} c_\omega \cdot \exp(2\pi i \omega t)\, d\omega$

c) $c_\omega = |c_\omega| \exp(i \gamma_\omega)$

$\exp(\alpha + \beta) = \exp(\alpha)\exp(\beta)$

$$= \int_{\omega \in \mathbb{R}} |c_\omega| \exp\left(2\pi i \left(\omega t + \frac{\gamma_\omega}{2\pi}\right)\right) d\omega$$

b)

$$= \int_{\omega \geq 0} |c_\omega| \exp(2\pi i(\omega t - \varphi_\omega))\, d\omega + \int_{\omega < 0} |c_\omega| \exp(2\pi i(\omega t - \varphi_\omega))\, d\omega$$

$$= \int_{\omega \geq 0} |c_\omega| \exp(2\pi(\omega t - \varphi_\omega))\, d\omega + \int_{\omega > 0} |c_{-\omega}| \exp(2\pi i(-\omega t - \varphi_{-\omega}))\, d\omega$$

$c_{-\omega} = \overline{c_\omega}$

$\varphi_{-\omega} = -\varphi_\omega$

$$= \int_{\omega \geq 0} |c_\omega| \exp(2\pi i(\omega t - \varphi_\omega))\, d\omega + \int_{\omega \geq 0} \overline{|c_\omega| \exp(2\pi i(\omega t - \varphi_\omega))}\, d\omega$$

$c + \bar{c} = 2\mathrm{Re}(c)$

$$= \int_{\omega \geq 0} 2\,\mathrm{Re}\left(|c_\omega| \exp(2\pi i(\omega t - \varphi_\omega)) \right) d\omega$$

$$= \int_{\omega \geq 0} |c_\omega| \cdot 2 \cdot \cos(2\pi(\omega t - \varphi_\omega))\, d\omega$$

$$\overset{a)}{=} \int_{\omega \geq 0} \sqrt{2}\, d_\omega \cos(2\pi(\omega t - \varphi_\omega))\, d\omega = 1)$$