# Cyclic sequences of $k$-subsets with distinct consecutive unions

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**Abstract**

In this paper, we investigate cyclic sequences which contain as elements all $k$-subsets of $\{0, 1, \ldots, n-1\}$ exactly once such that the unions of any two consecutive $k$-subsets of this sequences are pairwise distinct. Furthermore, if $Y$ is some prescribed subset of the power set of $\{0, 1, \ldots, n-1\}$, we require that all unions are in $Y$. In particular, we are interested in the case where $Y$ consists of all subsets of order having the same parity as $k$. Among others, we show the existence of such cyclic sequences for for $k = 2, 3, \ldots, 7$ and sufficiently large $n$. This kind of combinatorial problems is motivated from applications in combinatorial group testing. From our results, one obtains error detecting group testing procedures for items having the 2-consecutive positive property.

*Key words:* Distinct unions, $k$-subsets, group testing, middle two levels problem

## 1 Introduction

The following cyclic sequence $\mathcal{S}$ contains each 3-subset of the set $\{0, 1, 2, 3, 4, 5\}$ exactly once, where the first element of the sequence is considered to succeed the last one.

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$$\mathcal{S} = ( \begin{array}{l} 245, 025, 035, 013, 014, 124, 125, 235, 024, 034, \\ 135, 145, 023, 015, 045, 345, 234, 123, 012, \ 134 \ ) \end{array}$$

(Often, we will leave out the brackets in the set notation to improve readability. For example, the first set $\{2, 4, 5\}$ is simply written as 245.) The interesting property of this cyclic sequence is that the unions of any two consecutive 3-subsets are pairwise distinct. Indeed, one obtains the following list of unions:

$$( \begin{array}{l} 0245, \quad 0235, \quad 0135, 0134, 0124, 1245, 1235, 02345, \quad 0234, 01345, \\ 1345, 012345, 01235, 0145, 0345, 2345, 1234, \quad 0123, 01234, \ 12345 \ ) \end{array}$$

This is a typical example for the kind of combinatorial problem we are going to consider in this paper.

We now state the general problem introducing some notation. Let $\mathcal{P}(n) := 2^{[0:n-1]}$ denote the power set of $[0 : n-1] := \{0, 1, \dots, n-1\}$ and let $X \subset \mathcal{P}(n)$ be a subset with $m := |X|$. A *cyclic sequence* of the elements of $X$ is a sequence $\mathcal{S} = (x_1, \dots, x_m)$, $x_i \in X$, such that each element of $X$ appears exactly once in $\mathcal{S}$. The indices of the elements $x_i$ of $\mathcal{S}$ are considered modulo $m$, i. e., the element $x_1$ is the successor of the element $x_m$ and $x_{k+m} = x_m$ for $k \in \mathbb{Z}$. The sequence $\mathcal{S}$ is said to be a *cyclic sequence with distinct (consecutive) unions* (CSDU) if the unions $y_i := x_i \cup x_{i+1}$, $1 \leq i \leq m$, of any two consecutive elements of $\mathcal{S}$ are pairwise distinct. The class of all such sequences is denoted by CSDU$(X)$. In particular, we are interested in the case $X = \mathcal{P}(n, k)$, where $\mathcal{P}(n, k)$ denotes the set of all $k$-subsets of $[0 : n - 1]$. In this case we simply write CSDU$(n, k)$ := CSDU$(\mathcal{P}(n, k))$. For example, the cyclic sequence given above is an element in CSDU$(6, 3)$. One may further confine the choice of the unions by requiring that all unions $y_i$ are elements of some prescribed set $Y \subset \mathcal{P}(n)$. The subclass of CSDU$(X)$ of sequences with unions in $Y$ is denoted by CSDU$(X|Y)$. As above, we also write CSDU$(n, k|Y)$ if $X = \mathcal{P}(n, k)$. Finally, for the case $Y = \bigcup_{i=1}^{r} \mathcal{P}(n, \ell_i)$ for integers $\ell_1 < \ell_2 < \ldots < \ell_r$ we set CSDU$(n, k|\ell_1, \ell_2, \dots, \ell_r)$ := CSDU$(\mathcal{P}(n, k)|Y)$. For example, the following sequence is in CSDU$(6, 2|4)$:

$$\mathcal{S} = (05, 23, 01, 45, 13, 25, 04, 35, 24, 15, 03, 14, 02, 34, 12).$$

The concept of CSDUs, which to the best of the author's knowledge has not been considered before in the literature, is motivated by applications in combinatorial group testing. As we will see in Section 2, CSDUs lead to group testing procedures for items having the 2-consecutive positive property. Müller and Jimbo [7] give implicitly a construction of sequences in CSDU$(n, k)$ for all $n \geq 5$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. The case, where the unions are confined to some set $Y$, is much more difficult including many yet unsolved problems such as the middle two levels problem (see below). In this paper, we focus on the case $Y = \mathcal{P}_{(k \bmod 2)}(n)$, where $\mathcal{P}_0(n) \subset \mathcal{P}(n)$ consists of all sets of even order and $\mathcal{P}_1(n) \subset \mathcal{P}(n)$ consists of all sets of odd order. This case is of particular interest in view of the group testing application since it leads to group testing procedures which allow to detect up to one error (see Section 2). In the subsequent sections we give some general recursive constructions of CSDUs (Section 3) and a cyclic construction of CSDUs (Section 4). From the latter construction we obtain CSDUs for small parameters $n$ and $k$ which will serve as starting sequences for our recursions. As main result of this paper, we obtain sequences in $\mathrm{CSDU}(n, k|\mathcal{P}_{(k \bmod 2)}(n))$ for $k = 2, 3, \ldots, 7$ and sufficiently large $n$ (Section 5).

We close this introduction with a summary of related problems and known results. The concept of CSDUs fits into the framework of *combinatorial Gray codes* referring to any method for generating combinatorial objects so that successive objects differ in some pre-specified, usually small way. We refer to Savage [11] for an overview. The famous, yet unsolved *middle two levels problem* deals with the question whether there is a Hamiltonian cycle in the middle two levels $\mathcal{P}(2k+1, k)$ and $\mathcal{P}(2k+1, k+1)$ of the Boolean lattice $\mathcal{B}_{2k+1}$, i.e., the partially ordered set of all subsets of $[0 : n-1]$ ordered by inclusion. Obviously, such Hamiltonian cycles correspond exactly to the elements $\mathrm{CSDU}(2k+1, k|k+1)$. The largest value for which a Hamiltonian cycle is known to exist is $k = 15$, see Shields and Savage [12].

A generalization of the middle two levels problem is the *antipodal layers problem* which asks for a Hamiltonian cycle among $k$-sets and $(n-k)$-sets of $[0 : n-1]$, where two sets are joined by an edge if and only if one is a subset of the other (see, e. g., Hurlbert [6]). Note that any solution $\mathcal{S} = (x_1, \ldots, x_m) \in \mathrm{CSDU}(n, k|n-k)$ gives a solution $(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)$, $y_i = x_i \cup x_{i+1}$, of the corresponding antipodal layers problem. The converse, however, is in general not true since the condition that $y_i$ contains the two $k$-sets $x_i$ and $x_{i+1}$ is in general weaker than the condition that $y_i$ equals the union of $x_i$ and $x_{i+1}$. The antipodal layers problem has so far only been solved in case that $n \geq (3k + 1 + \sqrt{5k^2 - 2k + 1})/2$, see Chen [1].

Further notes and references can be found in Section 6. Concerning the group testing application references are given in the next section. We summarize the known results about CSDUs in the following theorem.

**Theorem 1.1** *(i) For any $n \in \mathbb{N}$ and $k$ with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ there is a cyclic sequence in $\mathrm{CSDU}(n, k)$ except for the parameters $n = 2, k = 1$ and $n = 4, k = 2$. (See Müller and Jimbo [7].)*

*(ii) There is a cyclic sequence in $\mathrm{CSDU}(2k+1, k|k+1)$ for $k = 1, \ldots, 15$. (See Shields and Savage [12].)*

## 2 Application to combinatorial group testing

To motivate the definition of CSDUs we give some short outline of some application to combinatorial group testing. Let $C = \{c_1, \ldots, c_m\}$ be a set of *items* and $\sigma : C \to \{0, 1\}$ a map indicating the *state* of each item. An item $c_i$ is said to be *positive* if $\sigma(c_i) = 1$, otherwise *negative*. In applications such as DNA library screening (in this case, the items are *clones*) one has the goal to determine the set of all positive items in $C$, where a method is given to *test* the state of each item (e.g., by some chemical analysis). To reduce the number of tests, one chooses a subset $P \subset C$, also denoted as *group* or *pool*, and tests all items of $P$ in one stroke. The state of a pool is *positive* if it contains at least one positive item, otherwise *negative*. This strategy is known as *group testing* which can be defined as the process of selecting pools and testing them to determine exactly which items are postive. A *nonadaptive* adaptive group testing procedure, where all pools are specified a priori without knowing the state of other pools, may be represented by some $n \times m$-incidence matrix $H = (h_{ij})$ over $\{0, 1\}$. Here, the columns of $H$ correspond with the $m$ items, the rows of $H$ correspond with the $n$ pools, and $h_{ij} = 1$ means that the $i$th pool contains the $j$th item $c_j$, $1 \le j \le m$, $1 \le i \le n$. For an overview of different group testing methods and some of their applications we refer to Du and Hwang [5].

Colbourn [2] considered the setting were the set $C$ is equipped with a linear order $c_i \prec c_{i+1}$, $1 \le i < m$, and has the *$d$-consecutive positive property*, i.e., the set of positive items is a consecutive set with respect to the ordering $\prec$ and contains at most $d$ items. He shows how to reduce the case $d \ge 2$ to the case $d = 2$. As is noted by Müller and Jimbo [7], one can distinguish up to any two consecutive positive items if all columns of $H$ as well as all vectors arising as bitwise OR-sum of two consecutive (or the first and last) columns of $H$ are pairwise distinct. Such matrices are also denoted as *cyclic 2-consecutive positive detectable matrices*. Furthermore, in view of the application it is desirable that each item has the same *replication number*, i.e., it appears the same number of times in the pools. In other words, all columns of the incidence matrix $H$ should have some fixed constant weight, say $k$.

Now, it is obvious that the incidence matrices $H$ of cyclic sequences $\mathcal{S} \in \mathrm{CSDU}(n, k)$ correspond in a one-to-one fashion to the cyclic 2-consecutive positive detectable matrices with $n$ rows, $m = \binom{n}{k}$ columns, and constant column weight $k$. For example, the incidence matrix $H$ of the sequence $\mathcal{S} \in \mathrm{CSDU}(6, 3)$ given in the introduction looks as follows:

$$
H = \begin{pmatrix}
1&1&1&0&0&0&1&1&0&0&1&1&0&1&1&1&0&0&0&0\\
1&0&0&0&1&1&0&0&1&1&0&1&0&0&1&1&1&0&0&1\\
0&0&1&1&0&0&0&1&0&1&1&0&1&0&0&1&1&1&0&1\\
1&1&0&0&0&1&1&1&1&0&0&0&1&0&0&0&1&1&1&0\\
0&0&0&1&1&1&1&0&0&0&1&1&0&1&0&0&0&1&1&1\\
0&1&1&1&1&0&0&0&1&1&0&0&1&1&1&0&0&0&1&0
\end{pmatrix}
$$

As is pointed out in [2,9,10], error correction capability of group testing procedures is essential in view of applications such as DNA library screening. Note that if the sets $x_i$ of some cyclic sequence $\mathcal{S} = (x_1, \ldots, x_m) \in \mathrm{CSDU}(n, k)$ and the corresponding unions $y_i = x_i \cup x_{i+1}$ are either all of even order or all of odd order, the set $\{x_i, y_i | 1 \leq i \leq m\}$ has minimum distance two. In view of this fact, the class $\mathrm{CSDU}(n, k|\mathcal{P}_{(k \bmod 2)}(n))$ is of particular interest. We summarize the result in the following theorem and refer for further details to Müller and Jimbo [7].

**Theorem 2.1** *Each $\mathcal{S} \in \mathrm{CSDU}(n, k)$ defines a group testing procedure for items having the 2-consecutive positive property with $n$ pools of size $\frac{k}{n} \cdot \binom{n}{k}$ and $m = \binom{n}{k}$ items, where each item appears in exactly $k$ pools. These group testing procedures are optimal in the sense that the number of items is maximal with respect to a fixed number of pools. If, in addition, $\mathcal{S} \in \mathrm{CSDU}(n, k|\mathcal{P}_{(k \bmod 2)}(n))$, the resulting group testing procedure can detect up to one error.*

## 3 Recursive constructions of CSDUs

As in the introduction, let $\mathcal{S} = (x_1, \ldots, x_m) \in \mathrm{CSDU}(n, k|Y)$ with $m = \binom{n}{k}$ and let $y_i = x_i \cup x_{i+1}$, $1 \leq i \leq m$, be the corresponding unions. Since $|y_i| > k$, $1 \leq i \leq m$, and since the $y_i$ are pairwise distinct, one obtains the following necessary condition.

**Lemma 3.1** *Let $\mathcal{S} \in \mathrm{CSDU}(n, k|Y)$, then $\binom{n}{k} \leq |\{y \in Y || y| > k\}|$.*

The permutation group $S_n$ of $[0 : n-1]$ induces in a natural way a group action on $\mathcal{P}(n)$ via $\sigma(\Gamma) := \{\sigma(\gamma) \mid \gamma \in \Gamma\}$ for $\Gamma \in \mathcal{P}(n)$ and $\sigma \in S_n$. The following obvious lemma will be applied in the subsequent constructions.

**Lemma 3.2** *Let $X$ and $Y$ be $S_n$-invariant subsets of $\mathcal{P}(n)$, then $\mathrm{CSDU}(X|Y)$ is invariant under the $S_n$-action, i.e., if $\mathcal{S} = (x_1, \ldots, x_m) \in \mathrm{CSDU}(X|Y)$ then $\sigma(\mathcal{S}) := (\sigma(x_1), \ldots, \sigma(x_m)) \in \mathrm{CSDU}(X|Y)$. Furthermore, for any $k \in [1 : m]$ one also has $(x_{k+1}, \ldots, x_m, x_1, \ldots, x_k) \in \mathrm{CSDU}(X|Y)$.*

Next we introduce some useful operators. For a $k$-subset $x \in \mathcal{P}(n, k)$, let $x^+$ be the $(k+1)$-subset of $[0 : n]$ by adding the element $n$, i.e., $x^+ := x \cup \{n\}$.

Furthermore, let $x^-$ denote the same $k$-subset $x$, however this time understood as a subset of $[0:n]$. These two operations may be repeatedly applied. For example, for the 2-subset $x = \{0,2\}$ of $[0:4]$ we obtain $x^+ = \{0,2,5\}$, $x^- = \{0,2\}$, $x^{++} = \{0,2,5,6\}$ and $x^{-+} = \{0,2,6\}$. Finally, we write $X^+ := \{x^+|x \in X\}$ and $X^- := \{x^-|x \in X\}$ for any $X \subset \mathcal{P}(n)$. The following lemma gives some easy recursive construction.

**Lemma 3.3** *Let $\mathcal{A} = (a_1, a_2, \ldots, a_r) \in \mathrm{CSDU}(n,k)$ with $r = \binom{n}{k}$ and $\mathcal{B} = (b_1, b_2, \ldots, b_s) \in \mathrm{CSDU}(n,k+1)$ with $s = \binom{n}{k+1}$. If $a_r \cup b_1 \neq b_s \cup a_1$ and $\{a_r \cup b_1, b_s \cup a_1\} \cap \{a_i \cup a_{i+1} | 1 \leq i \leq r-1\} = \emptyset$, then the cyclic sequence*

$$\mathcal{S} = ({a_1}^+, {a_2}^+, \ldots, {a_r}^+, {b_1}^-, {b_2}^-, \ldots, {b_s}^-)$$

*defines an element in* $\mathrm{CSDU}(n+1,k+1)$.

**Proof:** Obviously, $\mathcal{S}$ contains all $(k+1)$-subsets of $[0:n]$ exactly once. Furthermore, the conditions imposed on the $k$-subsets and $(k+1)$-subsets guarantee that all unions of two consecutive unions of elements in $\mathcal{S}$ are pairwise distinct. $\square$

For the next recursive construction, we need some technical condition on the cyclic sequences. We say that $\mathcal{S} = (x_1, \ldots, x_m) \in \mathrm{CSDU}(n,k|Y)$ has a *connector of size* $\ell$, $k+1 \leq \ell \leq 2k$, if there is some element $y \in Y$ of order $\ell = |y|$ such that

(1) $y$ does not appear among the unions $y_i = x_i \cup x_{i+1}$, $1 \leq i \leq m$, and
(2) there is some index $i \in [1:m]$ such that $x_i \subset y$, $|x_i \cup x_{i+1}| = \ell$, and $|y \setminus x_{i+1}| = 2(\ell - k)$.

We then also call $y$ an *$\ell$-connector* of $\mathcal{S}$.

**Theorem 3.4** *Let $Y$ and $Z$ be $S_n$-invariant subsets of $\mathcal{P}(n)$. Furthermore let $\mathcal{A} \in \mathrm{CSDU}(n,k|Y)$ be a cyclic sequence with $\ell$-connector $y$ and let $\mathcal{B} \in \mathrm{CSDU}(n,k+1|Z)$ be a cyclic sequence having at least one union of order $\ell+1$. Then there is some cyclic sequence $\mathcal{S} \in \mathrm{CSDU}(n+1,k+1|Y^+ \cup Z^-)$ having a connector of size $(\ell+1)$.*

**Proof:** Let $\mathcal{A} = (a_1, \ldots, a_r)$ and $\mathcal{B} = (b_1, \ldots, b_s)$ be as in the claim of the theorem. Since $\mathcal{A}$ has an $\ell$-connector and by applying Lemma 3.2, we may assume that $a_1 = \{0, \ldots, k-1\}$, $a_r = \{\ell-k, \ldots, \ell-1\}$, and $y = \{\ell-k, \ldots, 2\ell-k-1\}$, where $y \notin \{a_i \cup a_{i+1} \mid 1 \leq i \leq r\}$. Note that $|y \setminus a_1| = |\{k, \ldots, 2\ell-k-1\}| = 2(\ell-k)$. Similarly, we may assume that $b_1 = \{2\ell-2k-1, \ldots, 2\ell-k-1\}$ and $b_s = \{\ell-k-1, \ldots, \ell-1\}$. Then

(i) $a_r \cup b_1 = \{\ell-k, \ldots, 2\ell-k-1\} = y$,
(ii) $b_s \cup a_1 = \{0, \ldots, \ell-1\} = a_1 \cup a_r \neq y$, and

(iii) $\{a_r \cup b_1, b_s \cup a_1\} \cap \{a_i \cup a_{i+1} | 1 \leq i \leq r-1\} = \emptyset$.

Therefore, by Lemma 3.3, $\mathcal{S} = ({a_1}^+, \ldots, {a_r}^+, {b_1}^-, \ldots, {b_s}^-)$ is a cyclic sequence in CSDU$(n+1, k+1)$. Since the consecutive unions of $\mathcal{S}$ are obviously either in $Y^+$ or in $Z^-$, we obtain $\mathcal{S} \in$ CSDU$(n+1, k+1|Y^+ \cup Z^-)$. Finally, note that the element $z := {b_s}^- \cup {b_1}^- = \{\ell - k - 1, \ldots, 2\ell - k - 1\} \in Z^-$ is not among the consecutive unions of $\mathcal{S}$. Since $|z| = \ell + 1$, ${b_s}^- \subset z$, $|{b_s}^- \cup {a_1}^+| = |\{0, \ldots \ell - 1, n+1\}| = \ell + 1$, and $|z \setminus {a_1}^+| = |\{k, \ldots, 2\ell - k - 1\}| = 2(\ell - k) = 2(\ell + 1 - (k+1))$, $z$ is an $(\ell+1)$-connector of $\mathcal{S}$. $\square$

To apply the recursive construction from the last theorem we need starting sequences which will be constructed in the next section.

## 4 Cyclic constructions of CSDUs

The cyclic additive group $\mathbb{Z}_n$ of order $n$ operates on $[0 : n-1]$ by addition modulo $n$. This induces a group action on the set $\mathcal{P}(n, k)$ via $k + \Gamma := \{(k + \gamma) \bmod n \mid \gamma \in \Gamma\}$ for $\Gamma \in \mathcal{P}(n, k)$ and $k \in \mathbb{Z}_n$. We say that two $k$-sets $x$ and $x'$ are *equivalent* if they lie in the same $\mathbb{Z}_n$-orbit. A $\mathbb{Z}_n$-orbit is called *full* if it contains $n$ elements, otherwise it is called *short*. It is not hard to see that the length of a short orbit is a divisor of $n$.

Any $k$-subset $x \in \mathcal{P}(n, k)$ can be written in the form $x = \{j_1, j_2, \ldots j_k\}$ with $0 \leq j_1 < j_2 < \ldots j_k < n$. We associate to $x$ the sequence of differences $(d_1, d_2, \ldots, d_k)$ with $d_i = j_{i+1} - j_i$, $1 \leq i \leq k$, where indices are modulo $k$ and arithmetic is modulo $n$. Then $d_i > 0$ and $d_1 + d_2 + \ldots + d_k = n$. The sequence $(d_s, d_{s+1}, \ldots, d_k, d_1, \ldots, d_{s-1})$ is called a *cyclic shift* of $(d_1, d_2, \ldots, d_k)$ by $s$, $0 \leq s < k$. Two sequences of differences are called *equivalent* if one is a cyclic shift of the other. An equivalence class is written as $[d_1, d_2, \ldots, d_k]$ and denoted as *type* of $x$. Obviously, two $k$-sets are equivalent if and only if they are of the same type. A type is called *full* if it corresponds to a full orbit of $k$-sets, otherwise it is called *short*. For example, in the case $n = 5$ and $k = 2$ there are the two types $[1, 4]$ and $[2, 3]$ which are both full. The 2-subsets of type $[2, 3]$ are $\{0, 2\}$, $\{1, 3\}$, $\{2, 4\}$, $\{0, 3\}$, and $\{1, 4\}$. It is not hard to see that a short type corresponds to a short orbit of length $\ell$ if and only if the type is composed of $n/\ell$ copies of a contiguous subsequence of differences. For example, in the case $n = 6$ and $k = 4$ the type $[1, 2, 1, 2]$ corresponds to the short orbit of length $\ell = 3$ consisting of $\{0, 1, 3, 4\}$, $\{1, 2, 4, 5\}$, and $\{0, 2, 3, 5\}$. In particular, if $(n, k) = 1$, i.e., if $k$ is relatively prime to $n$, all orbits are full and the number of full orbits is $\binom{n}{k}/n$. Based on these observations, the following lemma gives a sufficient condition for a cyclic construction of CSDUs.

**Lemma 4.1** *Let $k$ be relatively prime to $n$ and $j := \binom{n}{k}/n$. Furthermore,* *let* $\mathcal{S}_0 = (x_1, x_2, \ldots, x_j, x_{j+1})$ *be a sequence of* $k$*-subsets of* $[0 : n-1]$*, and* $y_i = x_i \cup x_{i+1}$*,* $1 \le i \le j$*. Suppose the following conditions are fulfilled:*

*(i) The types of* $x_1, x_2, \ldots, x_j$ *are pairwise distinct, i.e.,* $x_1, x_2, \ldots, x_j$ *form a transversal of the* $\mathbb{Z}_n$*-action on* $\mathcal{P}(n, k)$*.*
*(ii)* $x_1$ *and* $x_{j+1}$ *are of the same type and* $x_{j+1} = x_1 + \delta$ *for some* $\delta$ *with* $(n, \delta) = 1$ *(here the addition is understood elementwise and modulo* $n$*).*
*(iii) The types of* $y_1, y_2, \ldots, y_j$ *are pairwise distinct and full.*

*Then* $\mathcal{S}_0$ *extends to some* $\mathcal{S} \in \mathrm{CSDU}(n, k)$*.*

**Proof:** Let $\mathcal{S}_0$ be a sequence as formulated in the lemma satisfying (i), (ii) and (iii). Furthermore, let $\mathcal{S}(i) := (x_1 + i, x_2 + i, \ldots x_j + i)$ for $0 \le i < n$, then one obtains the cyclic sequence

$$\mathcal{S} := (\mathcal{S}(0), \mathcal{S}(\delta), \mathcal{S}(2\delta), \ldots, \mathcal{S}((n-1)\delta))$$

of length $m = n \cdot j = \binom{n}{k}$ by concatenation. Since $(n, \delta) = 1$, all $\mathcal{S}(i)$, $0 \le i < n$, appear in this concatenation exactly once. From $(n, k) = 1$ and condition (i) it follows that all $k$-subsets of $\mathcal{P}(n, k)$ appear in $\mathcal{S}$ exactly once. From condition (ii) and condition (iii) it follows that all unions of consecutive $k$-subsets are pairwise distinct. $\square$

A sequence $\mathcal{S}_0$ satisfying the conditions (i), (ii), and (iii) of Lemma 4.1 will be called a *base sequence*. We give some examples.

**Example 4.2** (i) For $n = 5$ and $k = 2$ there are two full types $[1, 4]$ and $[2, 3]$ and for $\ell = 3$ two full types $[1, 1, 3]$ and $[1, 2, 2]$. One easily checks that $\mathcal{S}_0 = (01, 13, 12)$ defines a base sequence with $\delta = 1$ extending to a sequence in $\mathrm{CSDU}(5, 2|3)$.
(ii) For $n = 7$ and $k = 2$ there are the three full types $[1, 6]$, $[2, 5]$, $[3, 4]$. The sequence $\mathcal{S}_0 = (01, 24, 03, 12)$ defines a base sequence with $\delta = 1$ since the types of the unions are $[1, 1, 2, 3]$, $[1, 1, 3, 2]$, and $[1, 1, 1, 4]$ which are distinct and full. Therefore, in the resulting sequence $\mathcal{S} \in \mathrm{CSDU}(7, 2|4)$ no consecutive union is of type $[1, 2, 2, 2]$ or of type $[1, 2, 1, 3]$. Now, let $y = \{0, 1, 3, 5\}$. Then $y$ is of type $[1, 2, 2, 2]$, $\{0, 1\} \subset y$, $|\{0, 1\} \cup \{2, 4\}| = 4$, and $|y \setminus \{2, 4\}| = 4$, i.e., $y$ is a 4-connector of $\mathcal{S}$.
(iii) The sequence $\mathcal{S}_0 = (012, 267, 157, 247, 134, 457, 237, 567)$ defines a base sequence for some sequence in $\mathrm{CSDU}(8, 3|5)$.

**Theorem 4.3** *There is a cyclic sequence in* $\mathrm{CSDU}(n, 2|4)$ *if and only if* $n \ge 6$*. Furthermore, for all* $n \ge 7$ *there exists a cyclic sequence with a connector of size* 4*.*

**Proof:** By Lemma 3.1 there is no $\mathrm{CSDU}(n, 2|4)$ for $n \le 5$. The cases $n = 6$ and $n = 7$ are covered by the example in the introduction and by Example 4.2, respectively. Furthermore, the following sequence is in CSDU$(8, 2|4)$ with a connector of size 4 (consider, e.g., $y = \{0, 4, 5, 6\}$ with $x_i = \{5, 6\}$ and $x_{i+1} = \{2, 7\}$):

$$\begin{aligned}\mathcal{S} = (\,&06, 17, 26, 07, 36, 47, 16, 37, 56, 27, 46, 57, 01, 45,\\ &13, 25, 04, 35, 24, 15, 03, 14, 02, 34, 12, 05, 67, 23\ \ )\end{aligned}$$

We now consider the case for odd $n \geq 9$, i.e., $n = 2m + 1$ for $m \geq 4$, claiming that

$$\mathcal{S}_0 := (\{0, 1\}, \{2, 4\}, \{k, 2k + 3\} \text{ for } k = 0, 1, \dots, m - 3, \{2m, 0\}).$$

is a base sequence for CSDU$(n, 2|4)$ with $\delta = 2m$. The types of the first $m = \binom{n}{2}/n$ elements are $[1, n-1]$, $[2, n-2]$ and $[k+3, n-k-3]$ for $k = 0, 1, \dots, m-3$ which cover exactly all $m$ types for $n = 2m + 1$ and $k = 2$. Since $(n, k) = 1$, all types are full. Furthermore, the types of the first element $\{0, 1\}$ and last element $\{2m, 0\}$ of $\mathcal{S}_0$ coincide and $\delta = 2m$. For the unions we obtain the types $[1, 1, 2, n - 4]$, $[2, 1, 1, n - 4]$, $[1, k + 2, 2, n - (k + 5)]$ for $k = 0, 1, \dots, m - 4$, and $[m - 3, m, 3, 1]$. Since $m \geq 4$ and $(n, 4) = 1$, these types are pairwise distinct and full. Hence, $\mathcal{S}_0$ indeed defines a base sequence extending to some sequence $\mathcal{S} \in$ CSDU$(n, 2|4)$. Finally, note that $\mathcal{S}$ has a connector of size 4: Let $y = \{1, 2, 4, 5\}$ then $\{2, 4\} \subset y$, $|\{2, 4\} \cup \{0, 3\}| = 4$, and $|y \setminus \{0, 3\}| = 4$. Furthermore, the type $[1, 2, 1, n - 4]$ of $y$ differs from all types of the unions of $\mathcal{S}$, i.e., $y$ does not appear among the unions.

It remains to consider the case for even $n \geq 10$, i.e., $n = 2m$ for $m \geq 5$. Note that in the even case there is exactly one short orbit of type $[m, m]$ for $k = 2$. Nevertheless, we can use the same construction as for the odd case with a slight modification. Let

$$\mathcal{S}_0 = (\{0, 1\}, \{2, 4\}, \{k, 2k + 3\} \text{ for } k = 0, 1, \dots, m - 4, \{2m - 1, 0\}).$$

The types of the first $m - 1$ sets are $[1, n - 1]$, $[2, n - 2]$ and $[k + 3, n - k - 3]$ for $k = 0, 1, \dots, m - 4$ which cover exactly all full types. Furthermore, the types of the first element $\{0, 1\}$ and last element $\{2m - 1, 0\}$ coincide and $\delta = 2m - 1$. For the unions we obtain the types $[1, 1, 2, n - 4]$, $[2, 1, 1, n - 4]$, $[1, k+2, 2, n-(k+5)]$ for $k = 0, 1, \dots, m-5$, and $[m-4, m-1, 4, 1]$. One checks that all these types are full. Therefore, we can expanding $\mathcal{S}_0$ as in the proof of Lemma 4.1 resulting in a cyclic sequence $\mathcal{S}$ with distinct consecutive unions covering, however, only all 2-sets belonging to full orbits. We now extend $\mathcal{S}$ by inserting the sequence $(\{0, m\}, \{1, m + 1\}, \dots, \{m - 1, 2m - 1\})$, which consists of the missing 2-sets of the short orbit, between the elements $\{2, 4\}$ and $\{0, 3\}$ of $\mathcal{S}$. Then one checks that the type $[2, 2, m - 4, m]$ of the union $\{2, 4\} \cup \{0, m\}$, the type $[3, m - 4, m, 1]$ of the union $\{m - 1, 2m - 1\} \cup \{0, 3\}$, and the type $[1, m-1, 1, m-1]$ of the unions $\{k, m+k\} \cup \{k+1, m+k+1\}$ for $k = 0, \ldots, m-2$ are distinct and differ from all types of the unions of $\mathcal{S}$. Therefore, the so extended cyclic sequence defines indeed an element in $\mathrm{CSDU}(n, 2|4)$. Considering again the set $y = \{1, 2, 4, 5\}$ one sees as in the odd case that the sequence has a connector of size 4. $\square$

## 5 Some results and open problems

Combining the results of the last two sections, one can recursively construct CSDUs. For example, one obtains the following result.

**Corollary 5.1** *The class* $\mathrm{CSDU}(n, 3|\mathcal{P}_1(n))$ *is non-empty if and only if* $n \geq 8$. *In particular,* $\mathrm{CSDU}(n, 3|5)$ *is non-empty for* $n \geq 8$ *and there is a sequence with a* 5*-connector in* $\mathrm{CSDU}(n, 3|5)$ *for* $n \geq 9$.

**Proof:** By Lemma 3.1 there is no sequence in $\mathrm{CSDU}(n, 3|\mathcal{P}_1(n))$ for $n \leq 7$. By Theorem 4.3, there is a sequence with a 4-connector in $\mathrm{CSDU}(n, 2|4)$ for $n \geq 7$. Furthermore, Example 4.2 gives some sequence in $\mathrm{CSDU}(8, 3|5)$. Then, by Theorem 3.4, there is a sequence with a 5-connector in $\mathrm{CSDU}(9, 3|5)$. In the same fashion, one can apply Theorem 3.4 recursively for $n \geq 8$ to construct a sequence with a 5-connector in $\mathrm{CSDU}(n+1, 3|5)$ from a sequence with a 4-connector in $\mathrm{CSDU}(n, 2|4)$ and a sequence in $\mathrm{CSDU}(n, 3|5)$. $\square$

Now, using the CSDUs from Corollary 5.1 one can proceed in the same way to construct sequences with a 6-connector in $\mathrm{CSDU}(n, 4|6)$ for $n \geq n_4$ as long as one has a starting sequence in $\mathrm{CSDU}(n_4, 4|6)$ for some $n_4 \geq 10$. More generally, the following corollary holds.

**Corollary 5.2** *Suppose there is an increasing sequence* $n_3 \leq n_4 \leq n_5 \leq \ldots$ *of natural numbers* $n_k$, $k \geq 3$, *such that* $\mathrm{CSDU}(n_k, k|k+2)$ *is non-empty. Then there is a sequence (with a* $(k+2)$*-connector) in* $\mathrm{CSDU}(n, k|k+2)$ *for all* $n \geq n_k$ *(*$n > n_k$*).*

The hard part is to find starting sequences in $\mathrm{CSDU}(n_k, k|k+2)$ for suitable $n_k$. From Lemma 3.1 it follows that the smallest possible $n_k$ are given by $n_k = 2k+2$. The last corollary would then yield solutions for $\mathrm{CSDU}(n, k|k+2)$ for all parameters $n$ and $k$ satisfying the necessary condition implied by Lemma 3.1.

**Problem 5.3** *Is* $\mathrm{CSDU}(2k+2, k|k+2)$ *non-empty for all* $k \geq 2$*?*

This problem seems to be as hard as the middle two levels problem and even harder than the corresponding antipodal layers problem. We can relax the problem by either increasing the numbers $n_k$ or by enlarging the set $Y$ of admissible unions. In view of the group testing application a solution to the following problem would also give optimal solutions for group testing procedures with error detecting capability (see Theorem 2.1).

**Problem 5.4** *Is* $\mathrm{CSDU}(2k+2, k|\mathcal{P}_{(k \bmod 2)}(n))$ *non-empty for all* $k \geq 2$*? If yes, is there a sequence in each such* $\mathrm{CSDU}(2k+2, k|\mathcal{P}_{(k \bmod 2)}(n))$ *such that at least one consecutive union is of order* $k+2$*?*

So far, we have not yet succeeded to solve the last problem even though there seem to be a lot of freedom in the construction of such CSDUs. By a randomized algorithm we constructed with the help of a computer sequences for the following classes[1]: $\mathrm{CSDU}(11, 4|6)$, $\mathrm{CSDU}(12, 5|7, 9)$, $\mathrm{CSDU}(13, 5|7)$, $\mathrm{CSDU}(17, 6|8, 10)$ and $\mathrm{CSDU}(19, 7|9, 11)$ with at least one union having order $k+2$, respectively. This yields the following result.

**Corollary 5.5** *There is a sequence in* $\mathrm{CSDU}(n, k|\mathcal{P}_{(k \bmod 2)}(n))$ *for all* $n \geq n_k$ *for the following parameters* $k$ *and* $n_k$*:*

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $n_k$ | 6 | 8 | 11 | 12 | 17 | 19 |

Using a more involved strategy, e.g., similar to the one of Shield and Savage [12], one can easily extend this result by using a computer. The interesting question, however, is to find a general construction principle for infinitely many parameters $k$.

## 6 Final Remarks

In this paper we introduced the concept of cyclic sequences with distinct (consecutive) unions which is motivated from applications in nonadaptive group testing. In particular, we constructed such sequences for the case that all involved sets and unions have either even or odd order yielding group testing procedures which can detect up to one error. In our recursive construction, the hard part is to find suitable starting sequences. For the classes $\mathrm{CSDU}(2k+2, k|k+2)$ the problem is strongly connected to the yet unsolved middle two levels problem and the antipodal levels problem. However, for the less restrictive classes $\mathrm{CSDU}(n, k|\mathcal{P}_{(k \bmod 2)}(n))$ there seem to be quite a lot of freedom so that a general construction may be feasible.

To obtain group testing procedures with higher error detecting capability, one could try to start with some error correcting code $C$ of minimum distance $d$.

[1] The CSDUs can be obtained as text file from the first author via Email.

Then the question is, if one finds a partition $C = X\dot{\cup}Y$ such that there exists some cyclic sequence in $\mathrm{CSDU}(X|Y)$. Any such sequence would define a group testing procedures which can detect up to $d-1$ errors.

One further interesting problem is to prove lower bounds on the length of the longest cyclic sequence with distinct consecutive unions by constructing such sequences which not necessarily comprise all elements of $X$. For example, the middle two levels problem has been approached in this way: Shield and Savage [12] proved that there is a cycle in the middle two levels of the Boolean lattice $\mathcal{B}_{2k+1}$, having a length of at least 0.86 times the length of the corresponding optimal Hamiltonian cycle.

Note that the group testing problem, where one does not require the positives to be consecutive, is essentially different to the one discussed in this paper. The case, where one just assumes that the positive items are bounded by some number $d$, requires that the OR-sums of *any* $d$ (not necessarily distinct) columns of the group testing incidence matrix are pairwise distinct. This problem has lead to the concept of $d$-*disjunctive* matrices. For an overview and further references concerning these matrices we refer the reader to Du and Hwang [5] and Ngo and Du [9].

Finally, we want to mention another related concept of so-called $(d, f)$-cluttered orderings, which has been introduced and studied by Cohen and Colbourn [3,4] for various set systems such as the complete graphs and Steiner triple systems. For general set systems a $(d, f)$-cluttered ordering may be regarded as cyclic sequence consisting of all sets of the system such that the union of any $d$ consecutive sets contain at most $f$ points. For further details we refer to [3,4,8].

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