On the Numerical Instability of an LCMV Beamformer for a Uniform Linear Array

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***Abstract*—We analyze the conditions for numerical instability in the solution of a linearly constrained minimum variance (LCMV) beamformer with multiple directional constraints for a uniform linear array. An analytic expression is presented to determine the frequencies (for broadband signals such as speech) where the inverse term in the solution of the LCMV beamformer does not exist. Simulation results and power patterns are provided to further illustrate the problem. In addition, we investigate and discuss possible solutions to the problem.**

***Index Terms*—Array signal processing, beamforming, LCMV, microphone arrays.**

I. Introduction

MULTIPLE sensors have been commonly used to capture real world signals due to their ability to exploit the spatial diversity provided by sensors placed at different spatial locations. The domain of signal processing that aims to effectively process the output signals of these multiple sensors to extract a desired signal is known as array signal processing [1]. One of the most common array signal processing techniques, known as *beamforming*, aims to linearly combine the received signals to obtain the signal from a source(s) at a specific spatial location and get rid of the interference and noise components [1]–[4].

In speech enhancement, beamforming techniques have been successfully used to enhance the speech from a source positioned in a desired direction while suppressing the undesirable signal components due to the acoustic environment [2]. Beamformers in which multiple linear constraints are imposed (c.f. [5]) are known as linearly constrained minimum variance (LCMV) beamformer. The well known minimum variance distortionless response (MVDR) beamformer, is a special case of the LCMV beamformer with only a single constraint towards the desired source. In recent times, LCMV has gained popularity in the speech enhancement community [6]–[9]. In [10], it has been shown that theoretically, the LCMV beamformer can achieve perfect dereverberation and noise cancellation when the acoustic transfer functions (ATFs) between all sources (including interferences) and microphones are known. However, the ATFs are generally unknown and difficult to estimate accurately. In [6], it was shown that relative transfer function (RTF) based modified constraints gives the exact same LCMV filter.

One such method of formulating the constraints for the LCMV is to use the frequency-dependent array propagation vector/steering vector as directional constraint [11]–[13]. Such a formulation of the directional constraints for an LCMV beamformer is especially attractive when using a uniform linear array (ULA). For a ULA, the frequency-dependent steering vector can be easily formulated with complex exponentials in terms of the source DOAs and inter-element distance.

The LCMV beamformer has been extensively studied in literature. In [1], a comprehensive study of the formulation, theory, and practical aspects of the LCMV beamformer is provided for both narrowband and broadband signals. In [2], the LCMV beamformer is studied from the perspective of its use in room acoustic environments with microphone arrays. In the aforementioned literature, it has been remarked that for the existence of the inverse term in the solution of the LCMV filter, the noise covariance matrix needs to be of full-rank, and the columns of the constraint matrix should be linearly independent. However, these conditions are generic for any mathematical operation involving matrix inversions, and the studies do not provide any further insight into the influence of the geometric setup on the existence of the inverse term.

In this letter, we study the solution of the LCMV beamformer for a ULA, with directional constraints given by the steering vector considering a far-field model. The study in this letter is done for the narrowband processing of speech signals in the short-time Fourier transform (STFT) domain. The results are valid also for other transform domains. We analyze the linear dependency of the columns of the constraint matrix and present an analytic expression to determine the frequency(s) where the LCMV suffers from numerical instability due to the non-existence of the inverse term. In a previous study [14], it was shown that when the angular separation between the desired and interfering source is small, the LCMV suffers from potential instabilities as it becomes difficult to switch the gain from zero to one. This translates into appearance of sidelobes and displacement of the maxima of the main beam away from the source location. However, it should be noted that under these conditions, the directional constraints are still satisfied. The analysis presented here shows that irrespective of the angular separation between the desired and the interfering source(s), there exists certain frequency(s) where the LCMV fails to provide a numerically stable solution, i.e., it fails to satisfy the constraints.

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Through the analysis, we highlight the relation between the frequency of instability and the source DOAs. In multi-microphone speech enhancement applications, a higher angular separation between the desired and interfering source(s) is preferred for better extraction of the desired source. The presented analysis shows however that for the LCMV beamformer with directional constraints, a higher angular separation leads to a lower frequency of instability. With some illustrative examples, we also provide an insight into the behavior and applicability of the beamformer beyond the spatial aliasing frequency. The provided analytic expression also circumvents the need to check for linear dependency of the columns of the constraint matrix for each frequency, which can be computationally expensive especially when dealing with broadband signals like speech. In addition to the analysis, we also discuss possible solutions to the problem.

The rest of this letter is organized as follows. In Section II, we present the mathematical framework within which the problem is formulated. Section III gives the analysis of the condition for non-existence of solution. In Section IV, we investigate and discuss possible fallback solutions to this problem. In Section V, we give some concluding remarks.

II. Problem Formulation

Consider a ULA with M sensor elements located at $\mathbf{d}_{1\ldots M}$. For each time-frequency instant, we assume that the sound field is composed of at most $L \leq M$ plane waves propagating in some noise field. The $M \times 1$ vector of received signals in the frequency domain, $\mathbf{y}(f) = [Y(f,\mathbf{d}_1)\ldots Y(f,\mathbf{d}_M)]^T$, is given by

$$\mathbf{y}(f) = \underbrace{\sum_{l=1}^{L} \mathbf{x}_l(f)}_{\mathbf{x}(f)} + \mathbf{x}_{\mathrm{v}}(f), \tag{1}$$

where $\mathbf{x}_l(f) = [X_l(f,\mathbf{d}_1)\ldots X_l(f,\mathbf{d}_M)]^T$ contains the microphone signals corresponding to the l-th plane wave, and $\mathbf{x}_{\mathrm{v}}(f)$ is the noise component. The sound pressure corresponding to the l-th plane wave, i.e., the directional sound $\mathbf{x}_l(f)$ is given by

$$\mathbf{x}_l(f) = \mathbf{a}(\theta_l, f) X_l(f,\mathbf{d}_1), \tag{2}$$

where $\theta_l(f)$ is the DOA of the l-th plane wave ($\theta = 90^\circ$ denotes the array broadside). For a ULA with omnidirectional microphones, assuming the first microphone as the reference and a far-field model, the m-th element of the steering vector $\mathbf{a}(\theta_l, f)$ is given by

$$a_m(\theta_l, f) = \exp\{-j\kappa(m-1)d\cos\theta_l\}, \tag{3}$$

where $j = \sqrt{-1}$, d is the distance between two neighboring microphones, and $\kappa = 2\pi f/c$ is the wavenumber.

Assuming the components in (1) to be mutually uncorrelated, the power spectral density (PSD) matrix of the microphone signals can be expressed as

$$\begin{aligned}\mathbf{\Phi}_{\mathbf{y}}(f) &= \mathrm{E}\{\mathbf{y}(f)\mathbf{y}^{\mathrm{H}}(f)\} && (4)\\ &= \mathbf{\Phi}_{\mathbf{x}}(f) + \mathbf{\Phi}_{\mathrm{v}}(f), && (5)\end{aligned}$$

where $\mathbf{\Phi}_{\mathbf{x}}(f)$ and $\mathbf{\Phi}_{\mathrm{v}}(f)$ are the PSD matrices of the direct sound and the noise components, respectively. In general, the aim of beamforming is to enhance the desired signal(s) while suppressing the interference and noise. Therefore, the desired signal can be given by

$$Z(f) = \sum_{l=1}^{L} Q^*(\theta_l, f) X_l(f,\mathbf{d}_1) \tag{6}$$

where $Q^*(\theta_l, f)$ is the direction dependent gain and $(\cdot)^*$ denotes the complex conjugate. This gain function can be defined based on the application. The estimate of the desired signal can be given by

$$\hat{Z}(f) = \mathbf{w}^{\mathrm{H}}(f)\mathbf{y}(f)$$

where $\mathbf{w}(f)$ contains the beamformer weights.

One of the well known beamforming methods is the LCMV beamformer. The LCMV beamformer provides a fixed gain to the L sound sources while minimizing the additive noise contribution at the beamformer output. Mathematically, it can be given by [15]

$$\mathbf{w}_{\mathrm{LCMV}}(f) = \arg\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\mathbf{\Phi}_{\mathrm{v}}(f)\mathbf{w} \tag{7}$$

subject to

$$\mathbf{w}^{\mathrm{H}}(f)\mathbf{A}(f) = \mathbf{q}^{\mathrm{H}}(f) \tag{8}$$

where $\mathbf{A}(f)$ is the constraint matrix containing the steering vectors corresponding to each source DOA and $\mathbf{q}(f)$ contains the direction dependent gains. The solution is given by

$$\mathbf{w}_{\mathrm{LCMV}}(f) = \mathbf{\Phi}_{\mathrm{v}}^{-1}\mathbf{A}(f)\left[\mathbf{A}^{\mathrm{H}}(f)\mathbf{\Phi}_{\mathrm{v}}^{-1}\mathbf{A}(f)\right]^{-1}\mathbf{q}(f). \tag{9}$$

For the existence of the inverse term in (9), it is necessary that $\mathbf{\Phi}_{\mathrm{v}}$ is full-rank and the columns of $\mathbf{A}(f)$ are linearly independent [1]. In this letter, we analyze the linear dependency of the columns of the constraint matrix $\mathbf{A}(f)$ and derive an analytic expression to determine the frequency at which the inverse term does not exist for a given set of DOAs.

III. Non-Existence of Solution

Let us first consider two arbitrary column vectors of the matrix $\mathbf{A}(f)$, denoted by $\mathbf{a}(\theta_s, f)$ and $\mathbf{a}(\theta_t, f)$ where $s, t \in \{1, \ldots, L\}$. In the following we denote these vectors as $\mathbf{a}_s$ and $\mathbf{a}_t$, respectively, for brevity. By the definition of the steering vectors given in (3), $\mathbf{a}_s, \mathbf{a}_t \in \mathbb{C}^{M\times 1}$. To determine whether these two vectors are linearly dependent, we use the Cauchy-Schwarz inequality

$$|\langle \mathbf{a}_s, \mathbf{a}_t \rangle| \leq \langle \mathbf{a}_s, \mathbf{a}_s \rangle^{1/2} \langle \mathbf{a}_t, \mathbf{a}_t \rangle^{1/2}, \tag{10}$$

where $\langle\cdot\rangle$ is the *inner product* of two vectors defined as $\langle \mathbf{a}_s, \mathbf{a}_t \rangle \equiv \mathbf{a}_t^{\mathrm{H}}\mathbf{a}_s$. The equality holds if and only if $\mathbf{a}_s$ and $\mathbf{a}_t$ are dependent such that the inverse term in (9) does not exist. Given the expression for the steering vector in (3), we can see that the right hand side (RHS) of (10) equals to M ($\sqrt{M}\cdot\sqrt{M} = M$), i.e., the number of microphones. The left hand side (LHS) can be expanded as

$$|\langle \mathbf{a}_s, \mathbf{a}_t \rangle| = |1 + \exp\{j\kappa\, d(\cos\theta_t - \cos\theta_s)\} + \cdots + \exp\{j\kappa(M-1)d(\cos\theta_t - \cos\theta_s)\}|. \quad (11)$$

Now, for the equality in (10) to hold, the terms of the RHS in (11) should sum to M. We can write the individual terms in the RHS of (11) as $\exp\{j\phi\} = \cos\phi + j\sin\phi$, where $\phi = \kappa(m-1)d(\cos\theta_t - \cos\theta_s), m \in \{1, \ldots, M\}$. Then, we can see that for the terms to sum up to M, the real part of the individual terms need to be equal to 1, i.e., $\cos\{\kappa d(\cos\theta_t - \cos\theta_s)\} = 1$ and the imaginary parts should be zero, i.e., $\sin\{\kappa d(\cos\theta_t - \cos\theta_s)\} = 0$. Therefore, the argument of each individual exponential should satisfy the condition

$$\kappa d(\cos\theta_t - \cos\theta_s) = 2n\pi, \quad n \in \mathbb{N}^0. \quad (12)$$

Here, we consider the term for $m = 2$ only as the subsequent terms are the integer multiples of this term. It can be seen that $n = 0$ corresponds to the trivial case where $\theta_s = \theta_t$, which in practical terms translates to assigning different gains to the same direction. Ignoring this case and simplifying (12) for $n = 1$, the lowest frequency where the inverse term does not exist is given by

$$f = \frac{c}{d\,|\cos\theta_t - \cos\theta_s|}. \quad (13)$$

The absolute value of the denominator is taken since only positive frequencies are considered. This equation can also be expressed in terms of the narrowband spatial aliasing frequency as

$$f = \frac{f_a}{\left|\sin\left(\frac{\theta_s+\theta_t}{2}\right)\sin\left(\frac{\theta_s-\theta_t}{2}\right)\right|}, \quad (14)$$

where f_a is the spatial aliasing frequency, given by $f_a = c/(2d)$.

Two notable observations can be made from the expression given in (14). Firstly, since the range of the DOAs is [0, 180], the denominator term lies between 0 and 1. Therefore,

$$f \geq f_a \quad \forall \quad \theta_s, \theta_t \in [0, 180], \quad (15)$$

with equality if and only if the two angles are 0 and 180. Secondly, for any combination of DOAs θ_s and θ_t, the inverse term in the solution to the LCMV beamformer does not exist at a certain frequency. In practice, if the angles are close enough, this phenomena is not encountered, since then $f \gg F_s/2$, i.e., this phenomena occurs at a frequency much higher than the Nyquist frequency. It should also be noted that this phenomenon can occur at multiple frequencies within the Nyquist frequency range. It can either occur for the multiple combinations of $s, t \in \{1, \ldots, L\}$ or, if $f \ll F_s/2$, then the integer multiples of f can possibly lie within the Nyquist frequency range.

IV. Illustration and Discussion

To illustrate the problem, we analyze LCMV power patterns for three different source configurations. For all configurations, we considered a ULA with inter-microphone distance of $d =$ 3 cm and a sampling frequency of 16 kHz. The noise field was considered to be homogeneous and spatially white, i.e., $\mathbf{\Phi}_{\mathrm{v}} = \sigma_v^2(f)\mathbf{I}$. in Fig. 1 the power patterns are plotted for the whole DOA range across all frequencies upto 8 kHz, i.e., the Nyquist frequency. The spatial aliasing frequency for this array geometry is $f_a = 5.72$ kHz.

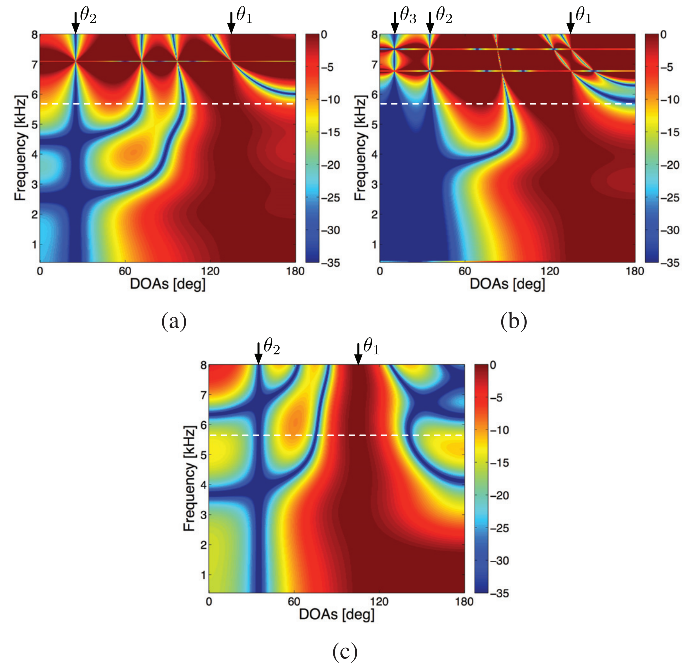

Fig. 1. Example LCMV power patterns for three different scenarios with the same inter-microphone distance $d = 3$ cm and sampling frequency $F_s =$ 16 kHz: (a) with 2 sources: $\theta_1 = 135°$, $\theta_2 = 25°$ (b) with 3 sources: $\theta_1 = 135°$, $\theta_2 = 35°$, $\theta_3 = 10°$ (c) with 2 sources: $\theta_1 = 105°$, $\theta_2 = 35°$. The dashed white line denotes the spatial aliasing frequency $f_a = 5.72$ kHz.

In Fig. 1(a), we consider 2 sources with DOAs $\theta_1 = 135°$ and $\theta_2 = 25°$, where θ_2 is the interference. Given the inter-microphone distance and the source DOAs, the frequency where the LCMV fails to provide a solution can be computed using (14). The computed frequency is $f = 7.09$ kHz. In Fig. 1(a), a line across the whole DOA range at that frequency can be seen. This line highlights the numerical instability of the provided solution. We can also observe that at this frequency the constraints are not satisfied, rather arbitrary gain at different DOAs can be observed. It is also worth noticing that this frequency lies above the narrowband spatial aliasing frequency. For the rest of the frequencies above the spatial aliasing frequency, the directional constraints are still satisfied.

In Fig. 1(b), we consider 3 source with DOAs $\theta_1 = 135°$, $\theta_2 = 35°$ and $\theta_3 = 10°$. In this scenario, we consider 2 undesired sources, θ_2 and θ_3. Here, we can see that there are two different frequencies where the line appears. This example highlights the possibility of multiple frequencies where the LCMV solution does not exist.

In Fig. 1(c), 2 sources were considered with DOAs $\theta_1 = 105°$ and $\theta_2 = 35°$, with the second source being the interference. The frequency where LCMV fails to give a solution was computed to be $f = 10.6$ kHz. It can be seen that this frequency lies beyond the Nyquist frequency, therefore in such a case this phenomena remains unnoticed when the beamformer is applied even above the spatial aliasing frequency.

From these power patterns, it can be seen that the LCMV beamformer with directional constraints can be safely applied

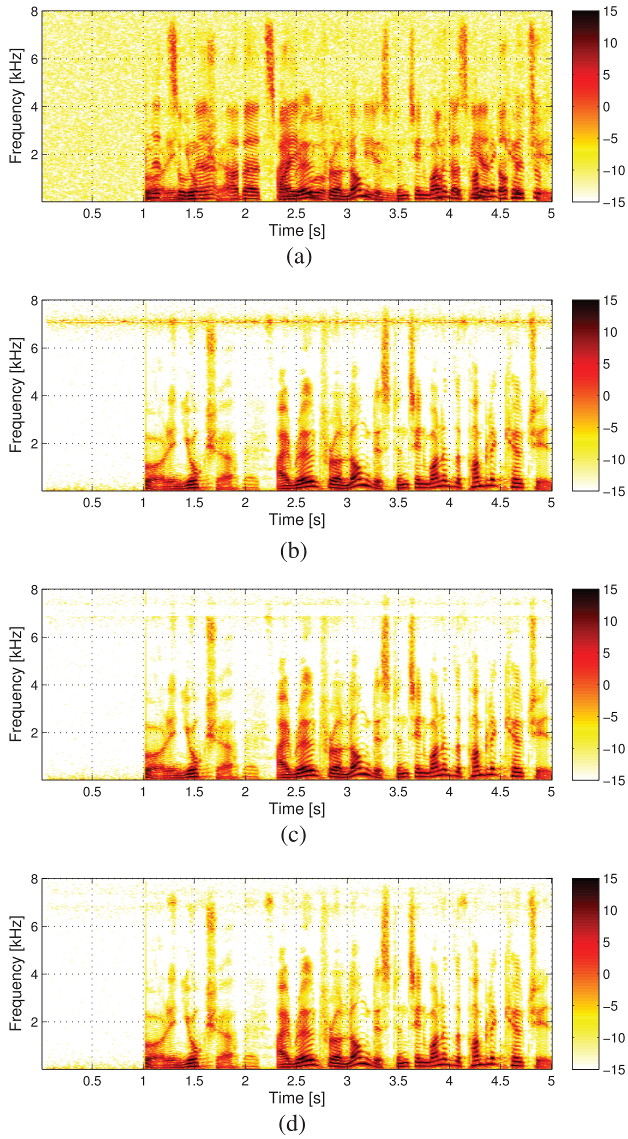

Fig. 2. Spectrograms for the example scenario considered in Fig. 1(a). (a) Mixture signal received at a reference microphone. (b) LCMV output. (c) LCMV output with modified constraints. (d) LCMV output with diagonal loading.

even above the spatial aliasing frequency except for a small range of frequencies around the frequency of instability.

We further illustrate the problem with a simulated experiment with speech signals. For the simulation, we consider the scenario for Fig. 1(a), with a ULA having $M = 4$ omnidirectional microphones. We consider an anechoic environment. White Gaussian noise was added to the microphone signals resulting in an average segmental signal-to-noise ratio of 25 dB. The signal received at the reference microphone is shown in Fig. 2(a). In Fig. 2(b), the output of the LCMV beamformer is presented. It can be clearly seen that, as in Fig. 1(a), a line appears at $f = 7.09$ kHz, where the inverse term in the solution of LCMV does not exist. In the following, we discuss and illustrate a few fallback solutions to the specified problem.

Given the expression in (14), the lowest frequency where the LCMV fails is dependent on the inter-microphone distance d and the source DOAs. For the implementation of the LCMV beamformer at the specific frequency, it is possible to modify d by selecting a subarray. However, a sub-array will always have a larger inter-microphone distance d than the original array and it can be clearly seen from the expression in (13) that this leads to a downward shift of the lowest frequency where the problem occurs.

To alleviate the problem, one might attempt to change the source DOAs in the implementation, thereby modifying the directional constraints of the beamformer. With such a method, the corresponding gains also need to be modified to satisfy the original constraints. However, this is theoretically not possible, since the basic reason the LCMV fails is that we assign different gains to the directions that exhibit the same spatial frequency. The LCMV implementation with the modified constraints for the considered experiment is shown in Fig. 2(c). The modified main beam direction was set at $\theta_1 = 120^\circ$ while keeping the null constant. It can be seen that though the line due to the instability is gone, for the frequency bins where this solution is applied, all signal components are completely suppressed.

One of the most common methods in literature to solve such ill-posed problems is *regularization*, where additional information is introduced in order to mitigate the problem [16]. Here, we investigate the simple regularization method of diagonal loading. In the solution of LCMV an extra term is added to the inverse term, which can be written as

$$\mathbf{w}_{\text{LCMV}}(f) = \mathbf{\Phi}_{\text{v}}^{-1}\mathbf{A}(f)\left[\mathbf{A}^{\text{H}}(f)\mathbf{\Phi}_{\text{v}}^{-1}\mathbf{A}(f) + \alpha\mathbf{I}\right]^{-1}\mathbf{q}(f), \tag{16}$$

where α is the real valued diagonal loading level. To ensure that effective regularization is applied when the instability occurs, based on the method presented in [17], we propose to compute α using

$$\alpha = \lambda \frac{1}{L} \text{tr}\left\{\mathbf{A}^{\text{H}}(f)\mathbf{\Phi}_{\text{v}}^{-1}\mathbf{A}(f)\right\}, \tag{17}$$

where $\text{tr}\{\cdot\}$ is the trace operator and L is the the number of constraints. The scalar λ is a signal independent scalar which determines the contribution of the loading level. The result of LCMV implementation with this solution is shown in Fig. 2(d). Though the initial problem is mitigated, the frequencies where the problem occurred, still contains spurious signal components. To avoid the instability problem, an MVDR beamformer can also be implemented at the corresponding frequency. However, only one of the original constraints can be satisfied.

Therefore, we can see that the discussed solutions remove the artifact introduced in the LCMV output due to the numerical instability, however, to the best knowledge of the authors, there is no solution for the considered scenario that can result in satisfying all original directional constraints.

V. Conclusion

Analysis of the conditions for numerical instability of the LCMV beamformer with multiple directional constraints, for a ULA, was presented and an analytic expression was derived to determine the frequencies where the problem occurs. We also discussed some intuitive possible solutions, which manage to resolve the numerical instability issue, but are unable to satisfy the original directional constraints.

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